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Non-local Yang–Mills theory

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Abstract. The Yang–Mills theory assumes that the parameters of the internal SU(2) group are functions of space–time. This is generalized by making the parameters matrices in space–time. When these are diagonal we recover the original theory. Instead of transforming according to just the regular representation, we find that the gauge field contains all representations consistent with angular momentum conservation. A further novelty is a metric introduced to make bilinears invariant and similarities to chiral invariance are pointed out. A suggested physical motivation is the existence of the Planck length.

1. Introduction

The utility of the gauge method of introducing interactions (e.g. Yang and Mills 1954) is now well appreciated. The formalism is also an attractive one. In this paper we wish to indicate what happens when we try to extend the ideas in the direction of non-locality.

It is always understood that the restriction to local interactions is somewhat artificial and in the nature of a postulate of simplicity and convenience. Because there are so many ways of relaxing this postulate, most people are reluctant to embark on a discussion of any one particular non-local theory. The aim of the present investigation is, however, a modest one and is simply to discover what kinds of fields are introduced by a non-local gauge technique. Consideration along these lines have already been made by Toro (1965). However, our conclusions seem to differ from his.

2. Basic ideas

We do not wish to go into details about the basic gauge method. These can conveniently be found in the review article by Adamski (1962).

The system we consider is that of a field $\varphi(x)$, which is a space–time scalar and which also belongs to a representation of an internal symmetry group. For simplicity we shall take this latter to be SU(2). The necessary algebra is then provided by the Wigner–Racah calculus. We shall then label φ as $\varphi_m^{(j)}$, where j refers to the 'spin-j' representation of dimension $2j+1$. The index m, running from $+j$ to $-j$, labels components inside this representation. Sometimes we shall combine the space–time variable x and the internal index m into a single one, as does De Witt (1964). Thus

$$\varphi_m^{(j)}(x) \equiv \varphi_m^{(j)} \qquad \varphi_{m'}^{(j)}(x') \equiv \varphi_{m'}^{(j)} \text{ etc.}$$

Internal indices† are raised and lowered by the internal metric C^{nm}

$$\varphi^m = C^{nm}\varphi_n$$

and we also employ the summation convention in the following form

$$\sum_m \int \varphi^m(x)\varphi_m(x)\,\mathrm{d}x = \varphi^m\varphi_m = \varphi_n C^{nm}\varphi_m.$$

† We could include the space–time index, which is 'raised and lowered' by means of a delta function.

Further, it is sometimes convenient to use a matrix notation. $\varphi_m^{(j)}(x)$ is then considered to be the representative of a vector and we can then write

$$\varphi_m^{(j)}(x) = \langle mx|\varphi, \qquad \tilde{\varphi}_m(x) = \tilde{\varphi}|mx\rangle$$

and

$$\varphi^m\varphi_m = \tilde{\varphi}\tilde{C}\varphi.$$

In this expression C is a matrix (or operator) with components

$$\langle mx|C|ny\rangle = C^{nm}\delta(x-y).$$

It is advantageous to have a variety of notations at hand depending on what aspect of the theory we wish to bring out. Sometimes a calculation is best performed with a minimum of clutter, and then a matrix type of notation is called for. In certain circumstances, however, e.g. when objects such as $3j$ symbols occur, a symbolic or matrix-like notation is, in general, a drawback.

Under an element of SU(2), φ transforms according to

$$\varphi(x) \to \exp(i\Lambda_i J^i)\varphi(x) = \varphi'(x) \tag{1}$$

where the J^i are the angular momentum matrices. Let us work with the infinitesimal form of (1); then the change in φ is given by

$$\delta_0\varphi_m(x) \equiv \varphi_m'(x) - \varphi_m(x) = i\Lambda_i[J^i]_m^{\cdot\,n}\varphi_n(x).$$

We now generalize this by making the Λ_i, which up to now have been constants, matrices (or operators) in space–time; i.e. we write

$$\delta_0\varphi_m(x) = i\int \Lambda_i(x, x')[J^i]_m^{\cdot\,m'}\varphi_{m'}(x')\,dx'. \tag{2}$$

The standard, local, gauge theory results on choosing Λ_i of diagonal form:

$$\Lambda_i(x, x') = \Lambda_i(x)\delta(x-x').$$

The original motivation of Yang and Mills for introducing position-dependent parameters $\Lambda_i(x)$ was the expectation that one should be able to perform the symmetry transformations at each space–time point independently. Can we then find a motivation for (2), which equation says that the new field at the point x depends on the values of the old field at *all* space–time points? A purely formal justification for discussing (2) is that it represents the widest generalization of (1), within the limits of linear, homogeneous transformations.

A vaguer but more physical motivation is to be found in the significance of the so-called 'Planck length',

$$L^* = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1{\cdot}6\times 10^{-33}\,\text{cm}$$

which constitutes a sort of limit beyond which the concepts of measurement and, hence, observable have no meaning. As De Witt (1964) remarks, the fundamental significance of this is not clear. Perhaps it means that fields cannot be described as functions defined on a Riemannian manifold. For Wheeler (e.g. 1964, 1968) it means that, at the L^* level, geometry fluctuates or resonates between many configurations and topologies, and it is only in the large that a single space–time manifold emerges. Speculatively we might imagine that these fluctuations will prevent us from making independent symmetry transformations at points separated by less than L^*, and that (2) will constitute a sort of phenomenological description of this. Of course, in this case the off-diagonal parts of Λ_i will only be appreciable for $x - x' \lesssim L^*$. Despite the apparent insignificance of this non-locality, it is a principle which is at stake here and once the possibility of non-local Λ_i is admitted the gauge theory becomes richer, i.e. less restrictive (and therefore, possibly, less useful). Let us then proceed with the formalism following from (2), which equation we write in the compressed way (see Birch and Dowker 1969)

$$\delta_0\varphi = \mathrm{i}\Lambda_i J^i\varphi. \tag{3}$$

If the theory is invariant under (1) it will not be so under (2), unless fields other than φ are introduced. One way of doing this is through the idea of the covariant derivative. From (2), the derivative of φ changes by

$$\delta_0\partial_\mu\varphi_m = \mathrm{i}\int \partial_\mu^x\Lambda_i(x, x')[J^i]_m^{\cdot\,m'}\varphi_{m'}(x')\,\mathrm{d}x'$$

or, formally, using the momentum operator P_μ with

$$\langle mx|P_\mu|m'x'\rangle = \mathrm{i}\delta^m_{\cdot\,m'}\partial_\mu\delta(x-x')$$

we have

$$\delta_0 P_\mu\varphi = \mathrm{i}P_\mu\Lambda_i J^i\varphi. \tag{4}$$

In order to facilitate comparison with the local and constant Λ_i cases, we write (4) as follows:

$$\delta_0 P_\mu\varphi = \mathrm{i}[P_\mu, \Lambda_i]J^i\varphi + \mathrm{i}\Lambda_i J^i P_\mu\varphi. \tag{5}$$

Now

$$[P_\mu, A] = \mathrm{i}\overset{+}{\partial}_\mu A$$

where

$$\langle mx|\overset{+}{\partial}_\mu A|m'x'\rangle \equiv (\partial_\mu^x + \partial_\mu^{x'})A_m^{\cdot\,m'}(x, x'). \tag{6}$$

If $A(x, x')$ is diagonal, $A(x, x') = a(x)\delta(x-x')$,

$$[P_\mu, A] = \mathrm{i}\partial_\mu a 1$$

where 1 is the unit operator in space–time.

In the local and constant parameter case (equation (1)) Λ_i is proportional to 1, and so the first term on the right-hand side of (5) is zero, i.e. $\partial_\mu\varphi$ transforms like φ. If the parameters are functions (local or non-local) this is no longer the case, and one seeks for a generalized derivative†, which we shall denote by $K_\mu\varphi$, that does transform like φ; i.e.

$$\delta_0 K_\mu\varphi = \mathrm{i}\Lambda_i J^i K_\mu\varphi. \tag{7}$$

K_μ is an operator in both internal space and space–time. For comparison purposes we shall put K_μ equal to $-\mathrm{i}[P_\mu + A_\mu]$ and shall consider A_μ as the (potentials of the) introduced gauge field which undergoes transformations in order to cancel out the first term on the right-hand side of (5).

From (7) and (3), we have

$$\delta_0 K_\mu\varphi = (\delta_0 K_\mu)\varphi + K_\mu\delta_0\varphi = (\delta_0 K_\mu)\varphi + \mathrm{i}K_\mu\Lambda_i J^i\varphi \equiv \mathrm{i}\Lambda_i J^i K_\mu\varphi$$

and so

$$\delta_0 K_\mu = \mathrm{i}[\Lambda_i J^i, K_\mu]$$

or

$$\delta_0 A_\mu = \overset{+}{\partial}_\mu\Lambda_i J^i + \mathrm{i}[\Lambda_i J^i, A_\mu]. \tag{8}$$

We see from (5) that making Λ_i functions introduces extra terms proportional to J^i into the derivative. It might thus seem that the compensating field A_μ need only

† cf. Birch and Dowker (1969). In this paper the sign of P_μ is reversed.

be proportional to J^i also. In the local case this is correct, but in the non-local situation a new effect arises due to the non-commutativity of space–time operators.

If A_μ were proportional to J^i then, for reasons of covariance, we would want $\delta_0 A_\mu$ also to be so proportional. However, the commutator term in (8) in fact introduces terms involving the product J^iJ^j, which in general cannot be expressed as a linear combination of J^i. If this process is continued, A_μ will have to be a sum of terms proportional to J^i, J^iJ^j, $J^iJ^jJ^k$, ... , up to a product of $2j$ generators.

Instead of using this series of products as what is, in effect, a complete set of $(2j+1)$-square basis matrices we shall employ, equivalently, the $3j$ symbols which are the matrix elements of Racah's (1951) quantities $u_{(k)}^{\,q}$

$$\langle jm|u_{(k)}^{\,q}|jm'\rangle = \begin{pmatrix} j & q & m' \\ m & k & j \end{pmatrix}, \qquad k = 0, 1, \dots, 2j.$$

Thus we write

$$A_\mu = \sum_k A_{\mu\ q}^{\ (k)} u_{(k)}^{\,q}. \tag{9}$$

Further, the generators J^i are proportional to $u_{(1)}^{\,i}$ and so let us absorb the constants into the Λ_i.† Equation (8) then reads

$$\delta_0 A_\mu = \overset{+}{\partial}_\mu \Lambda_i u_{(1)}^{\,i} + \mathrm{i} \sum_k [\Lambda_i u_{(1)}^{\,i}, A_{\mu\ q}^{\ (k)} u_{(k)}^{\,q}]$$

or, separating the space–time and internal operators,

$$\delta_0 A_\mu = \overset{+}{\partial}_\mu \Lambda_i u_{(1)}^{\,i} + \tfrac{1}{2}\mathrm{i} \sum_k ([\Lambda_i, A_{\mu\ q}^{\ (k)}]\{u_{(1)}^{\,i}, u_{(k)}^{\,q}\} + \{\Lambda_i, A_{\mu\ q}^{\ (k)}\}[u_{(1)}^{\,i}, u_{(k)}^{\,q}]).$$

The analysis is now purely one in Wigner–Racah calculus.

The expression for the product of two u's given by Racah (1951) allows us to write $\delta_0 A_\mu$ as

$$\delta_0 A_\mu = \sum_{k'=0}^{2j} \delta_0 A_{\mu\ q'}^{\ (k')} u_{(k')}^{\,q'}$$

where

$$\begin{aligned}\delta_0 A_{\mu\ q'}^{\ (k')} = {} & \delta_1^{k'} \overset{+}{\partial}_\mu \Lambda_{q'} - \mathrm{i}\tfrac{1}{2} \sum_k^{\text{even}} [\Lambda_i, A_{\mu\ q}^{\ (k)}](-1)^{k+2j} \begin{Bmatrix} 1 & j & j \\ j & k & k' \end{Bmatrix} \begin{pmatrix} i & q & k' \\ 1 & k & q' \end{pmatrix} (2k+1) \\ & - \mathrm{i}\tfrac{1}{2} \sum_k^{\text{odd}} \{\Lambda_i, A_{\mu\ q}^{\ (k)}\}(-1)^{k+2j} \begin{Bmatrix} 1 & j & j \\ j & k & k' \end{Bmatrix} \begin{pmatrix} i & q & k' \\ 1 & k & q' \end{pmatrix} (2k+1).\end{aligned}$$

The summations are for $k+1-k'$ even and odd as indicated. The summations over k are restricted by the $3j$ symbols to the terms $k = k'-1$, k' and $k'+1$. The first and third of these are 'even' and the second 'odd' terms. Thus we find

$$\begin{aligned}\delta_0 A_{\mu\ q'}^{\ (k')} = {} & \delta_1^{k'} \overset{+}{\partial}_\mu \Lambda_{q'} + \mathrm{i}\tfrac{1}{2}[\Lambda_i, A_{\mu\ \ q}^{\ (k'-1)}](-1)^{k'+2j} \begin{Bmatrix} 1 & j & j \\ j & k'-1 & k' \end{Bmatrix} \begin{pmatrix} i & q & k' \\ 1 & k'-1 & q' \end{pmatrix} (2k'-1) \\ & + \mathrm{i}\tfrac{1}{2}[\Lambda_i, A_{\mu\ \ q}^{\ (k'+1)}](-1)^{k'+2j} \begin{Bmatrix} 1 & j & j \\ j & k'+1 & k' \end{Bmatrix} \begin{pmatrix} i & q & k' \\ 1 & k'+1 & q' \end{pmatrix} (2k'+3) \\ & - \mathrm{i}\tfrac{1}{2}\{\Lambda_i, A_{\mu\ q}^{\ (k')}\}(-1)^{k'+2j} \begin{Bmatrix} 1 & j & j \\ j & k' & k' \end{Bmatrix} \begin{pmatrix} i & q & k' \\ 1 & k' & q' \end{pmatrix} (2k'+1).\end{aligned} \tag{10}$$

† When we do this we must be careful to remember that the parameters depend (in a trivial way) on the representation, i.e. on j.

From this structure we can deduce what we already know, namely that taking A_μ proportional to just $u_{(1)}$ is not consistent. If we assume that $A_\mu{}^{(k')}_{q}$ is proportional to $\delta^{k'}_1$, equation (10) shows that $\delta_0 A_\mu{}^{(k')}_{q'}$ contains terms proportional to $\delta^{k'}_0$ and $\delta^{k'}_2$ as well as to $\delta^{k'}_1$. Continuing this procedure, i.e. building up a finite transformation†, we see that, finally, all terms $k = 0, \ldots, 2j$ will be needed in the expansion (9) if the formalism is to be covariant.

In the local case (when the Λ_i and $A_\mu{}^{(k)}_{q}$ are diagonal) the second and third terms on the right-hand side of (10) vanish, and assuming that only $k = 1$ contributes in (9) is consistent because $\delta_0 A_\mu{}^{(k')}_{q'}$ is now zero unless $k' = 1$. However, it is also consistent to have *any* selection of terms in (9) but only the $k = 1$ term is *forced* upon us (by the derivative in (10)). Thus we would reject all fields with k not equal to unity as having nothing to do with the gauge method. We thus arrive at the standard result that the compensating gauge field transforms according to the regular (here, 'spin'-1) representation of the internal symmetry group. The last term in (10) gives the change in $A_\mu{}^{(k')}_{q'}$ under a constant-parameter transformation and since the $3j$ symbol $\begin{pmatrix} i & q & k' \\ 1 & k' & q' \end{pmatrix}$ is proportional to the generators J^i in the k' representation we see that $A_\mu{}^{(k')}$ also belongs to this representation.

In the non-local case, as we have said, we cannot get away with having just one term in the expansion (9) but need them all. Therefore, if we use the gauge method for the non-local situation *first* and *then* take the local limit we are still left with all the fields with k' not equal to unity, whereas if we start with the local case right from the beginning these fields do not really force themselves upon us. We get two different situations depending on whether we take the local limit before or after applying the gauge technique. As an example we can take the case when the symmetry group is that associated with isotopic spin (Yang and Mills 1954). If the basic fields have isotopic spin equal to $\frac{1}{2}$ ($j = \frac{1}{2}$), e.g. nucleons, then taking the local limit afterwards yields gauge fields with isotopic spin of both 1 and 0, whereas only the spin-1 field occurs if the local limit is taken first. If we take j equal to one, e.g. Σ-particles, then the gauge field has isotopic spin 2, 1 and 0 components, again unless the local limit is taken first.

The actual relation between these gauge fields and any physical particles is unclear and is probably a consequence of detailed dynamics (e.g. Schwinger 1962). We, therefore, make no specific predictions, although it is perhaps not unreasonable to suppose that the group multiplet structure would not be altered by the dynamics. For this reason we briefly mention here that for the SU(3) internal symmetry group a non-local gauge method applied to, say, quark basic fields yields a nonet, $8 \oplus 1$, rather than an octet, 8, of vector gauge fields, e.g. Ne'eman (1961). Assuming that the dynamics can give these fields mass, we can now make the relevant physical statement that vector mesons do occur in nines, rather than eights, as in the $8 \oplus 1$ set (ρ, ω, K^*, ϕ).

3. Development of formalism

All the gauge-field formalism can now be repeated with the expression (9) for the potential. Thus the field strength $R_{\mu\nu}$ defined by

$$R_{\mu\nu} = 2\mathrm{i} K_{[\mu} K_{\nu]}$$

$$= 2[\overset{+}{\partial}_{[\mu} A_{\nu]} + \mathrm{i} A_{[\mu} A_{\nu]}]$$

can be expanded in the $u_{(k)}$ as

$$R_{\mu\nu} = \sum_k R_{\mu\nu}{}^{(k)}_{q} u_{(k)}^{q}$$

† We could have used finite transformations from the start.

where

$$R_{\mu\nu}{}^{(k)}_{q} = 2\,\overset{+}{\partial}_{[\mu}A_{\nu]}{}^{(k)}_{q} + \mathrm{i}\sum_{k_1,k_2}^{\text{even}} [A_{\mu}{}^{(k_1)}_{q_1}, A_{\nu}{}^{(k_2)}_{q_2}](-1)^{k_1+k_2+2j}\begin{Bmatrix} k_1 & j & j \\ j & k_2 & k \end{Bmatrix}\begin{pmatrix} q_1 & q_2 & k \\ k_1 & k_2 & q \end{pmatrix}(2k+1)$$
$$+\mathrm{i}\sum_{k_1,k_2}^{\text{odd}} \{A_{\mu}{}^{(k_1)}_{q_1}, A_{\nu}{}^{(k_2)}_{q_2}\}(-1)^{k_1+k_2+2j}\begin{Bmatrix} k_1 & j & j \\ j & k_2 & k \end{Bmatrix}\begin{pmatrix} q_1 & q_2 & k \\ k_1 & k_2 & q \end{pmatrix}(2k+1). \tag{11}$$

The summations are again restricted, as indicated, to k_1+k_2-k even and odd. Further, k_1, k_2, k must satisfy the triangle inequality. In the local case only the odd summation survives and (11) reduces to

$$r_{\mu\nu}{}^{(k)}_{q} = \overset{+}{\partial}_{[\mu}a_{\nu]}{}^{(k)}_{q} + \mathrm{i}\sum_{k_1,k_2}^{\text{odd}} a_{\mu}{}^{(k_1)}_{q_1}a_{\nu}{}^{(k_2)}_{q_2}(2k+1)(-1)^{k_1+k_2+2j}\begin{Bmatrix} k_1 & j & j \\ j & k_2 & k \end{Bmatrix}\begin{pmatrix} q_1 & q_2 & k \\ k_1 & k_2 & q \end{pmatrix} \tag{12}$$

where we have put

$$R_{\mu\nu}{}^{(k)}_{q}(x, x') = r_{\mu\nu}{}^{(k)}_{q}(x)\delta(x-x')$$

$$A_{\mu}{}^{(k)}_{q}(x, x') = a_{\mu}{}^{(k)}_{q}(x)\delta(x-x').$$

If only $k = 1$ contributes to (9) then only $k_1 = 1$, $k_2 = 1$ contributes to (12) and hence $R_{\mu\nu}{}^{(k)}_{q}$ is zero unless $k = 1$, because of the 'odd restriction'. This is the standard result of local gauge theory. When all terms in (9) are present then so are all terms in (12) unless some further assumption is made. In the usual theory such assumptions generally take the form of the vanishing of the covariant derivative of the fundamental objects of the theory. Let us see how this comes about in the *local* theory.

Implicit in our calculation so far has been the assumption that the $3j$ symbols do not vary from point to point. This is not the most general possibility. In effect it corresponds to choosing at each point internal space bases which are related by SU(2) transformations and not by those of GL(2, C) (the most general case). A formalism covariant under the first type of transformations will not necessarily be so under the second.

We now prove the following results. If the covariant derivative of the $3j$ symbol vanishes, $R_{\mu\nu}$ can be expressed in terms of just the generators, i.e. $R_{\mu\nu}{}^{(k)}_{q}$ is zero unless $k = 1$. If, further, the $3j$ symbols are constant, $A_{\mu}{}^{(k)}_{q}$ is also zero except for $k = 1$. The calculation is simple. We note that the quantity

$$H(x) = \begin{pmatrix} m_1 & m_2 & m_3 \\ \dot{j}_1 & \dot{j}_2 & \dot{j}_3 \end{pmatrix}\varphi_{m_1}^{j_1}(x)\varphi_{m_2}^{j_2}(x)\varphi_{m_3}^{j_3}(x)$$

is a space–time as well as an internal scalar and so its covariant derivative is the ordinary derivative,

$$\nabla_\mu H = \partial_\mu H.$$

If we expand both sides of this equation by the distributive rule and use

$$\nabla_\mu\varphi = \partial_\mu\varphi + \mathrm{i}a_\mu\varphi$$

we find, if

$$\nabla_\mu \begin{pmatrix} m_1 & m_2 & m_3 \\ \dot{j}_1 & \dot{j}_2 & \dot{j}_3 \end{pmatrix} = 0$$

$$\mathrm{i}\{[a_\mu]_{m_1}^{\cdot\, m_1'}\delta_{m_2}^{\cdot\, m_2'}\delta_{m_3}^{\cdot\, m_3'} + \delta_{m_1}^{\cdot\, m_1'}[a_\mu]_{m_2}^{\cdot\, m_2'}\delta_{m_3}^{\cdot\, m_3'} + \delta_{m_1}^{\cdot\, m_1'}\delta_{m_2}^{\cdot\, m_2'}[a_\mu]_{m_3}^{\cdot\, m_3'}\}\begin{pmatrix} m_1 & m_2 & m_3 \\ \dot{j}_1 & \dot{j}_2 & \dot{j}_3 \end{pmatrix}$$
$$= \partial_\mu \begin{pmatrix} m_1' & m_2' & m_3' \\ \dot{j}_1 & \dot{j}_2 & \dot{j}_3 \end{pmatrix}. \tag{13}$$

If the right-hand side of this equation vanishes then it implies that the a_μ are proportional to the generators. (This follows from the numerical invariance of the $3j$ symbols under SU(2).) If the right-hand side does not vanish we proceed by acting on (13) with ∂_ν and using (13) again on the resulting left-hand side, and obtain,

$$\{[r_{\mu\nu}]_{m_1}^{\cdot\ m_1'}\delta_{m_2}^{\cdot\ m_2'}\delta_{m_3}^{\cdot\ m_3'}+\delta_{m_1}^{\cdot\ m_1'}[r_{\mu\nu}]_{m_2}^{\cdot\ m_2'}\delta_{m_3}^{\cdot\ m_3'}+\delta_{m_1}^{\cdot\ m_1'}\delta_{m_2}^{\cdot\ m_2'}[r_{\mu\nu}]_{m_3}^{\cdot\ m_3'}\}$$
$$\times\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3\end{pmatrix}=0$$

as the integrability condition for (13). This equation means that $r_{\mu\nu}$ is proportional to the generators.

With this calculation in mind we return to the non-local case and we must now take into account a circumstance we have not mentioned before (cf. Birch and Dowker 1969). Consider the bilinear quantity

$$I=\psi_m C^{mn}\varphi_n=\tilde{\psi}\tilde{C}\varphi=\chi^\dagger\varphi.$$

Under the local SU(2) transformation

$$\delta_0\varphi=\mathrm{i}\lambda_i J^i\varphi,\qquad \delta_0\psi=\mathrm{i}\lambda_i J^i\psi\quad\text{or}\quad\delta_0\chi=\mathrm{i}\lambda_i J^i\chi$$

I is unchanged because of the relation

$$CJ^i+\tilde{J}^iC=0,\qquad \tilde{J}^i=J^{i*}.$$

However, under the non-local transformation (3) I is not invariant. In fact we have

$$\delta_0 I=\tilde{\psi}\tilde{C}(\Lambda_i-\tilde{\Lambda}_i)J^i\varphi=\chi^\dagger(\Lambda_i-\tilde{\Lambda}_i)J^i\varphi.$$

For $\delta_0 I$ to vanish, Λ_i must be symmetric, $\Lambda_i=\tilde{\Lambda}_i$, which is not true in general, and so we introduce a compensating 'field' M and redefine I by

$$I=\tilde{\psi}M\varphi=\chi^\dagger\tilde{C}^{-1}M\varphi.$$

Note that M has two 'upstairs' indices.

We assume that M, like A_μ, is a matrix in internal space as well as in space–time. If we liked, we could now take ψ and φ to belong to different representations of SU(2). We shall not, however, pursue this possibility here.

If $\delta_0 I$ is to be zero, M must transform too and the change in M is given by

$$\delta_0 M=-\mathrm{i}(\tilde{J}^i\tilde{\Lambda}_i M+M\Lambda_i J^i). \tag{14}$$

It is not possible to choose an M of the simple form

$$M=M'C$$

where M' is an SU(2) scalar, because $\delta_0 M$ will not have such a form, unless it vanishes. Thus we again take M to be a series

$$M=\sum_k M_q^{(k)}\tilde{C}u_{(k)}^{q}. \tag{15}$$

$\delta_0 M_q^{(k)}$ can be found in exactly the same way as was $\delta_0 A_\mu{}^{(k)}$ earlier.

Extending these ideas to trilinear quantities we consider, in particular, an object of the form†

$$L=\psi^{(j)}_{\ l}\varphi^{(j)}_{\ m}\chi^{(k)}_{\ n}N^{lmn}_{(j)(j)(k)}$$

† For $k=0$, N coincides with M.

where we have reverted to De Witt's notation. For convenience we now drop the j, k labels whenever possible without confusion.

We require $\delta_0 L$ to vanish, and find the condition for this to be

$$\delta_0 N^{l'm'n'} = -\{\Lambda_l^{\cdot l'}\delta_m^{\cdot\, m'}\delta_n^{\cdot\, n'} + \delta_l^{\cdot l'}\Lambda_m^{\cdot\, m'}\delta_n^{\cdot\, n'} + \delta_l^{\cdot l'}\delta_m^{\cdot\, m'}[\Lambda(k)]_n^{\cdot\, n'}\}N^{lmn}$$

where, e.g.

$$\Lambda_l^{\cdot l'}N^{lmn} \equiv \sum_i \int \mathrm{d}x\Lambda_i(x, x')[J^i]_l^{\cdot l'}N^{lmn}(x, y, z).$$

It is inconsistent to have $N^{lmn}(x, y, z)$ equal to just a function of x, y and z multiplied by the $3j$ symbol and, again, we should expand it in terms of the $u^{(k')}$, for *all* k' and not just $k' = k$,

$$N_{l\,.\,n}^{\cdot m\cdot}(x, y, z) = \sum_{k'} N_{nq}^{(k')}(x, y, z)[u_{(k')}^{\;q}]_l^{\cdot\, m}.$$

We now calculate the covariant derivatives of the M and N quantities, firstly M. Since $\tilde{\psi}M\varphi$ is a constant and an SU(2) scalar, its covariant derivative vanishes

$$\nabla_\mu(\tilde{\psi}M\varphi) = 0.$$

We expand this equation by the distributive rule and assume that the covariant derivative of M is of the form

$$\nabla_\mu M = \overset{+}{\partial}_\mu M + \mathrm{i}\tilde{B}_\mu M + \mathrm{i}MB_\mu \tag{16}$$

where the B_μ are to be found. We find, if surface terms are thrown away

$$\tilde{\psi}[\tilde{A}_\mu M + MA_\mu + \tilde{B}_\mu M + MB_\mu] = 0$$

i.e.

$$B_\mu = -A_\mu. \tag{17}$$

Similarly, the covariant derivative of N^{lmn} is found to be

$$\nabla_\mu N^{lmn} = \overset{+}{\partial}_\mu N^{lmn} - \mathrm{i}[A_{\mu l'}^{\;\cdot\, l}\delta_{m'}^{\cdot\, m}\delta_{n'}^{\cdot\, n} + \delta_{l'}^{\cdot\, l}A_{\mu m'}^{\;\cdot\, m}\delta_{n'}^{\cdot\, n} + \delta_{l'}^{\cdot\, l}\delta_{m'}^{\cdot\, m}[A_\mu(k)]_{n'}^{\cdot\, n'}]N^{l'm'n'}$$

where

$$\overset{+}{\partial}_\mu N^{lmn} = (\partial_\mu^x + \partial_\mu^y + \partial_\mu^z)N^{lmn}(x, y, z).$$

The generalization to an object $N^{m_1\,m_2\,m_3}_{(j_1)(j_2)(j_3)}$ is obvious.

Thus, at least, making the covariant derivatives vanish implies nothing about the composition of A_μ. Further, integrability does give relations between $R_{\mu\nu}$ and N^{lmn}, but these are not as algebraically tractable as those from (13).

4. Interpretation and conclusion

We can think of M as providing a sort of metric† in SU(2) representation space cum space–time. In the case when Λ_i is symmetric we have, from (14), that $\delta_0 M$ vanishes if M takes the form,

$$\langle mx|M|ny\rangle = C^{mn}\delta(x-y). \tag{18}$$

Under these conditions we say that M is numerically invariant. If Λ_i is not symmetric, M is not numerically invariant. This situation can be compared with that for Lorentz

† This is only a formal concept and implies no deep or 'unified' relation between internal space and space–time.

and general coordinate transformations. Under the former, the metric $\eta_{\mu\nu}$ is numerically invariant while if general coordinate transformations are allowed then the metric is, in general, of the curvilinear type $g_{\mu\nu}$ and is not numerically invariant. We know that if the metric $g_{\mu\nu}$ is such that it can be transformed everywhere to the $\eta_{\mu\nu}$ form by a coordinate transformation, then there is no true gravitational field present, according to Einstein's theory. Arguing by analogy, we say that the metric M will represent a compensating field, which is not a 'true' one if there exists a transformation (3) that reduces M to the form (18). In general we need *all* the terms in the expansion (15). Again, as in the case of the A_μ, if we take the local limit *after* applying the gauge method *all* these terms will remain.

What we have in mind is that the Langrangian for the basic fields will contain a mass term, quadratic or bilinear in the fields, which will have to be modified by the insertion of the metric M if it is to be invariant under (3). For example, let us again choose the case of $j = \frac{1}{2}$, then in the sum (15) we have only the terms $k = 0$ and $k = 1$ and we can use for the $u_{(k)}$ the unit matrix 1 and the Pauli matrices τ^i. Thus we would have

$$\tilde{C}^{-1}M = M_0 1 + M_i\tau^i \tag{19}$$

and the mass term is modified in the following way

$$\kappa^2\varphi^\dagger\varphi \to \kappa^2 M_0\varphi^\dagger\varphi + \kappa^2 M_i\varphi^\dagger\tau^i\varphi.$$

The first term of the right-hand side of this expression can still be interpreted as a mass term, while the second shows the typical interaction of an isovector particle field. If we adjusted the space–time properties of the basic fields φ we could arrange for this isovector particle to be a pseudoscalar one, and hence we see that the non-local gauge method provides not only for vector particles (e.g. A_μ) but also for (pseudo) scalar ones. Thus we have a motivation for the pion as a 'gauge particle'.

Just as the M can be considered as a metric, so the A_μ can be looked upon as a kind of 'connection'. In general we would expect no relation between M and A_μ. However, by analogy with Riemannian geometry, we shall make what is probably the most restrictive assumption, namely that the covariant derivative of M vanishes, and we hope that the resulting more specific formalism possesses some physical significance.

In order that our discussion should bear some resemblance to reality we are going to consider the pion–nucleon system. Our basic fields are then two Pauli two-spinors φ, χ which are also two component iso-spinors. φ and χ combine to give a Dirac field. The non-interacting, original Langrangian takes the form

$$\mathscr{L} = \mathrm{i}\{\varphi^\dagger\sigma^\mu\partial_\mu\varphi + \chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi - \kappa(\chi^\dagger\varphi - \varphi^\dagger\chi)\}.$$

We now employ the non-local gauge method to the isospin group SU(2), under which φ and χ transform in the same way, and then take the local limit. The new Lagrangian now is

$$\mathscr{L} = \tfrac{1}{2}\mathrm{i}\{\varphi^\dagger(m+m^\dagger)\sigma^\mu\nabla_\mu\varphi + \chi^\dagger(m+m^\dagger)\bar{\sigma}^\mu\nabla_\mu\chi\} - \kappa\mathrm{i}(\chi^\dagger m\varphi - \varphi^\dagger m^\dagger\chi) \tag{20}$$

where we have added the Hermitian conjugate and integrated by parts to achieve this form. In this Lagrangian, m is the two-by-two SU(2) matrix metric given by

$$m = m_0 1 + \mathrm{i}\boldsymbol{m}.\boldsymbol{\tau}$$

where m_0 and $\boldsymbol{m}$ are real, which choices are governed by the desire to have an interaction with the correct inversion properties. The covariant derivative ∇_μ has the form

$$\nabla_\mu = \partial_\mu + \mathrm{i}a_\mu$$

with a_μ given by

$$a_\mu = \mathrm{i}a_{\mu 0}1 + \boldsymbol{a}_\mu.\boldsymbol{\tau}$$

with $a_{\mu 0}$ and $\boldsymbol{a}_\mu$ real.

The condition that $\nabla_\mu m$ should vanish yields

$$\partial_\mu m = \mathrm{i}(ma_\mu - a_\mu{}^\dagger m)$$

or

$$\partial_\mu m_0 = -2a_{\mu 0} m_0 \tag{21}$$

and

$$\partial_\mu \boldsymbol{m} = -2a_{\mu 0}\boldsymbol{m} - 2\boldsymbol{m} \wedge \boldsymbol{a}_\mu. \tag{22}$$

We now consider the mass term of (20). Since m is not unitary, in general, it is not possible to transform it away into the kinetic part of the Lagrangian by a canonical transformation of the nucleon fields. From (21) and (22) we have that

$$|\boldsymbol{m}| = \alpha m_0, \qquad \alpha = \text{constant},$$

and so

$$mm^\dagger = m_0{}^2 + \boldsymbol{m} \,.\, \boldsymbol{m} = (1+\alpha^2)m_0{}^2.$$

Thus, only in the case that m_0 and $|\boldsymbol{m}|$ are constant can we make m unitary, and we might as well then put m_0 equal to unity straightaway and then identify $\boldsymbol{m}$ with $(g/\kappa)\boldsymbol{\pi}$ so that the mass term becomes

$$\mathscr{L}_2 = +\kappa\mathrm{i}(\varphi^\dagger\chi - \chi^\dagger\varphi) + g\boldsymbol{\pi} \,.\, (\chi^\dagger\boldsymbol{\tau}\varphi + \varphi^\dagger\boldsymbol{\tau}\chi) \tag{23}$$

while the kinetic part is

$$\mathscr{L}_1 = \mathrm{i}(\varphi^\dagger\sigma^\mu(\partial_\mu + \mathrm{i}a_\mu)\varphi + \chi^\dagger\bar{\sigma}^\mu(\partial_\mu + \mathrm{i}a_\mu)\chi)$$

with a_μ given by the solution of (22) and (21),

$$a_{\mu 0} = 0, \qquad \boldsymbol{a}_\mu = -\frac{1}{2}\frac{1}{\boldsymbol{\pi}\,.\,\boldsymbol{\pi}}\boldsymbol{\pi} \times \partial_\mu \boldsymbol{\pi}. \tag{24}$$

If we wish to eliminate the non-derivative coupling of (23) in favour of derivative couplings we can perform a further unitary transformation. This, of course, gives the well-known equivalence theorem (e.g. Nelson 1941, Dyson 1948, Case 1949, Drell and Henley 1952). We have not yet finished, for the further unitary single-valued transformation,

$$\varphi \to T\exp\left(-\mathrm{i}\int^x a_\mu \,\mathrm{d}x^\mu\right)\varphi$$

eliminates the a_μ term from $\mathscr{L}_1$. This follows from the fact that the field $r_{\mu\nu}$ calculated from an a_μ of the form (24) is zero. The final Lagrangian is then

$$\mathscr{L} = \mathrm{i}(\varphi^\dagger\sigma^\mu\partial_\mu\varphi + \chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi) + g\boldsymbol{\pi} \,.\, (\chi^\dagger\boldsymbol{\tau}\varphi + \varphi^\dagger\boldsymbol{\tau}\chi) + \mathrm{i}\kappa(\varphi^\dagger\chi - \chi^\dagger\varphi)$$

plus a possible 'pion'-only part. Of course this theory cannot be considered seriously since $|\boldsymbol{\pi}|$ is constant. It corresponds to the σ model of Gell-Mann and Levy (1960) in the limit of infinite f. In this limit the SU(2) $\otimes$ SU(2) group reduces to the isospin SU(2) group. The SU(2) $\otimes$ SU(2) group is that of independent isospin transformations on φ and χ. Even in the local case it is necessary to introduce a 'compensating' field in order that the nucleon mass term should be invariant. This is the approach of Gürsey (1960) and Chang and Gürsey (1967). The motivation for introducing the field is not, however, of the normal Yang–Mills type.†

The similarity of the Gürsey approach to chiral invariance to the present non-local considerations suggests that there is some deeper connection between the two.

† By 'normal' we mean the usual technique of making the parameters functions of position in *space–time*. We could extend this and make the parameters functions of all, or any, quantum numbers including parity. In this extended sense we might say that the Gürsey method is of the Yang–Mills type.

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